SUPERFIELD DESCRIPTION OF A SELF-DUAL SUPERGRAVITY a la MACDOWELL-MANSOURI

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Abstract

Using MacDowell-Mansouri theory, in this work, we investigate a superfield description of the self-dual supergravity *a la* Ashtekar. We find that in order to reproduce previous results on supersymmetric Ashtekar formalism, it is necessary to properly combine the supersymmetric field-strength in the Lagrangian. We extend our procedure to the case of supersymmetric Ashtekar formalism in eight dimensions.

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1.- Introduction

The MacDowell-Mansouri formalism [1]-[2] (see also Refs. [3]-[6]) is one of the most interesting attempts to describe gravity as a gauge field theory. The basic assumption in such a formalism is to consider a curvature defined in terms of a $SO(1,4)-$valued one-form connection A^{AB} which is broken into a $SO(1,3)-$valued one-form connection A^{ab} and one-form tetrad field $A^{4a} = e^a$. The corresponding proposed action is quadratic in the reduced action with the skew symmetric $\varepsilon-$symbol acting as a contracting object. An interesting aspect of this action is that it is reduced to the Einstein-Hilbert action when the $\varepsilon-$symbol is associated with a four dimensional spacetime. Indeed, since a curvature is a two-form the quadratic action gives a four-form, which can be contracted with an $\varepsilon-$symbol of the form $\varepsilon^{\mu\nu\alpha\beta}$. These observations mean that the MacDowell-Mansouri theory is intrinsically a four dimensional gauge theory of gravity. On the other hand, one of the most interesting alternatives for quantum gravity is provided by the Ashtekar theory [7] which is also intrinsically a four dimensional theory of gravity. So a natural step further was to see if the MacDowell-Mansouri formalism and the Ashtekar theory were related. It turns out that a connection between these two theories was achieved by Nieto, Socorro, and Obregón [8] whom proposed a MacDowell-Mansouri type action, but with the curvature replaced by a self-dual curvature. The interesting thing of this approach is that by dropping the Euler and Pontrjagin topological invariants the Nieto-Socorro-Obregón action leads to the Ashtekar formalism via the Jacobson-Smolin-Samuel action [9].

Moreover, it has been shown that the Nieto-Socorro-Obregón action can be extended to the case of a self-dual supergravity in four dimensions [10]. However, this treatment refers only to the case of pure supergravity. If one desires to introduce interactions some problems arise -similar to the ones presented in the original theory of the MacDowell-Mansouri theory for supergravity. In particular, the need of introducing a metric of the spacetime weakens the original idea of seeing the self-dual supergravity as a gauge theory. Thus, one is forced to look for an alternative for supersymmetrizing the self-dual gravity with the hope that the interactions appear more naturally and in the spirit of a gauge theory.

As it is known, a superfield formalism [11] provides one of best mechanisms to supersymmetrisize a physical system, and is a particularly useful tool in the construction of interacting Lagrangians. In this work we explore the possibility of making a description of the self-dual supergravity a la MacDowell-Mansouri theory in terms of the superfield formalism. We prove that our description

leads exactly to the same action as the one proposed by Nieto-Socorro-Obregón for the case of the self-dual supergravity in four dimensions.

Another source of motivation for the present work may arise from the following observations. Traditionally, the fact that the Ashtekar theory requires a four dimensional scenario for its formulation, it is seen as a key feature over the alternative for quantum gravity based on a superstring theory which requires a higher dimensional spacetime background. However, if such Jacobson-Samuel-Smolin action is considered as a BF theory, then a possible generalization to higher dimensions of the Ashtekar formalism could be possible [12]. But in this BF approach the self-duality concept is lost. An alternative has been proposed by Nieto [13] in order to promote the Ashtekar formalism in the sense of self-duality of higher dimensions. In this case, the octonion structure [14]-[16] becomes the mathematical key tool. In fact, by using an octonion algebra in Ref. [13] it has been shown that it makes sense to consider an Ashtekar formalism in eight dimensions. It turns out that this generalization opens the possibility to consider Ashtekar formalism in twelve dimensions either by the reduction $10{+}2 \rightarrow (1{+}3){+}(1{+}7)$ or by the prescription $10{+}2 \rightarrow (2{+}2){+}(0{+}8)$ [17]. The case $(1+3)+(1+7)$ is particularly interesting because establishes that the Ashtekar formalism is not completely unrelated to the string theory or $M-$theory. While the case $(2+2)+(0+8)$ is interesting because $(2+2)$ and $(0+8)$ are exceptional signatures. Both proposals $10+2 \rightarrow (1+3)+(1+7)$ and $10+2 \rightarrow (2+2)+(0+8)$ have been developed using only a bosonic degree of freedom and it is the goal of this work to direct the first steps toward a supersymmetrization of such proposals.

Technically, the article is organized as follows: In section 2, we give a short sketch of the self-dual MacDowell-Mansouri theory. In section 3, we briefly describe the supersymmetric MacDowell-Mansouri theory. In section 4, we explicitly show that such a supersymmetric version of the MacDowell-Mansouri theory can be accomplished via superfield formalism. Finally, in section 5, we outline a possible application of our results to the case of an Ashtekar formalism in a spacetime of $2+10-$dimensional spacetime.

2.- Self-dual MacDowell-Mansouri theory

The theory of MacDowell-Mansouri is a pure gauge theory in $3+1$ dimensions, with the anti-de Sitter group $SO(3,2)$ as the gauge group. After breaking the original gauge group to $SO(3,1)$, the resultant $SO(3,1)$ gauge theory leads to the Einstein-Hilbert action, the cosmological constant term and the Euler topological invariant.

By convenience, let us recall MacDowell-Mansouri mechanism. Consider the MacDowell-Mansouri type action in a $1+3-$dimensional spacetime M^{1+3},

$$S=\int_{M^{1+3}} d^4x\varepsilon^{\mu\nu\rho\sigma}\mathcal{R}_{\mu\nu}^{\ ab}\mathcal{R}_{\rho\sigma}^{\ cd}\varepsilon_{abcd}. \tag{1}$$

Here, $\varepsilon^{\mu\nu\rho\sigma}$ and ε_{abcd} are the completely antisymmetric ε-symbols and $\mathcal{R}_{\mu\nu}^{\ ab}$ is given by

$$\mathcal{R}_{\mu\nu}^{\ ab}=F_{\mu\nu}^{\ ab}+\Sigma_{\mu\nu}^{\ ab}, \tag{2}$$

with

$$F_{\mu\nu}^{\ ab}=\partial_\mu A_\nu^{\ ab}-\partial_\nu A_\mu^{\ ab}+A_\mu^{\ ac}A_{\nu c}^{\ \ b}-A_\mu^{\ bc}A_{\nu c}^{\ \ a} \tag{3}$$

and

$$\Sigma_{\mu\nu}^{\ ab}=e_\mu^a e_\nu^b-e_\mu^b e_\nu^a. \tag{4}$$

where $A_\mu^{\ ab}$ is a $SO(1,3)$-connnection and e_μ^a is a tetrad field.

Substituting (2) into (1) we get

$$S=\int_{M^{1+3}}(T+K+C), \tag{5}$$

with

$$T=\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}^{\ ab}F_{\alpha\beta}^{\ cd}\varepsilon_{abcd}, \tag{6}$$

$$K=2\varepsilon^{\mu\nu\alpha\beta}\Sigma_{\mu\nu}^{\ ab}F_{\alpha\beta}^{\ cd}\varepsilon_{abcd}, \tag{7}$$

and

$$C=\varepsilon^{\mu\nu\alpha\beta}\Sigma_{\mu\nu}^{\ ab}\Sigma_{\alpha\beta}^{\ cd}\varepsilon_{abcd}. \tag{8}$$

It is not difficult to recognize that T leads to the Euler topological invariant, K determines the Einstein-Hilbert action and C gives the cosmological constant term.

In the case of the self-dual (antiself-dual) MacDowell-Mansouri theory the action (1) is extended in the following form

$$S=\int_{M^{1+3}} d^4x\varepsilon^{\mu\nu\rho\sigma+}\mathcal{R}_{\mu\nu}^{\ ab+}\mathcal{R}_{\rho\sigma}^{\ cd}\varepsilon_{abcd}, \tag{9}$$

where

$$
{}^{\pm}\mathcal{R}_{\mu\nu}^{\ ab} = \frac{1}{2}{}^{\pm}B_{cd}^{ab}\mathcal{R}_{\mu\nu}^{\ cd}. \tag{10}
$$

Here,

$$
{}^{\pm}B_{cd}^{ab} = \frac{1}{2}(\delta_{cd}^{ab} \pm i\varepsilon_{cd}^{ab}), \tag{11}
$$

where $\delta_{cd}^{ab} = \delta_c^a\delta_d^b - \delta_d^a\delta_c^b$ is a generalized delta.

We can again write (9) as in (5), namely

$$
S' = \int_{M^{1+3}} (T' + K' + C'), \tag{12}
$$

where T', K' and C' have the same form as T, K and C, but with $F_{\mu\nu}^{\ ab}$ and $\Sigma_{\mu\nu}^{\ ab}$ replaced by ${}^{+}F_{\mu\nu}^{\ ab}$ and ${}^{+}\Sigma_{\mu\nu}^{\ ab}$, respectively.

Using the identity

$$
{}^{+}B_{ef}^{ab+}B_{gh}^{cd}\varepsilon_{abcd} = 2(\varepsilon_{efgh} - i\delta_{efgh}), \tag{13}
$$

with $\delta_{efgh} = \eta_{eg}\eta_{fh} - \eta_{eh}\eta_{fg}$, we discover that T', K' and C' can be written as

$$
T' = \frac{1}{2}(\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}^{\ ab}F_{\alpha\beta}^{\ cd}\varepsilon_{abcd} - i\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}^{\ ab}F_{\alpha\beta}^{\ cd}\delta_{abcd}), \tag{14}
$$

$$
K' = (\varepsilon^{\mu\nu\alpha\beta}\Sigma_{\mu\nu}^{\ ab}F_{\alpha\beta}^{\ cd}\varepsilon_{abcd} - i\varepsilon^{\mu\nu\alpha\beta}\Sigma_{\mu\nu}^{\ ab}F_{\alpha\beta}^{\ cd}\delta_{abcd}), \tag{15}
$$

and

$$
C' = \frac{1}{2}(\varepsilon^{\mu\nu\alpha\beta}\Sigma_{\mu\nu}^{\ ab}\Sigma_{\alpha\beta}^{\ cd}\varepsilon_{abcd} - i\varepsilon^{\mu\nu\alpha\beta}\Sigma_{\mu\nu}^{\ ab}\Sigma_{\alpha\beta}^{\ cd}\delta_{abcd}). \tag{16}
$$

The first and the second term in T' corresponds to the Euler and Pontrjagin topological invariants respectively. The first term in K' leads to (7), while the second term in (15) vanishes identically after using the cyclical identities of $F_{\alpha\beta}^{\ cd}$. Similarly, the first term in C' corresponds to (8), while the second term of C' is identically zero. These results show that up to the Pontrjagin topological invariant the actions (1) and (9) are equivalents. It is worth observing that, despite ${}^{\pm}\mathcal{R}_{\mu\nu}^{\ ab}$ is a complex field, these results imply that, classically, the action (9) itself can be reduced to a real quantity (see Refs. [8]-[10] for details). At the quantum level, however, this is not true, but one can use Ashtekar mechanism [7] in order to develop a consistent canonical quantization out of the action (9).

3.- Supersymmetric MacDowell-Mansouri theory

The supersymmetric version of MacDowell-Mansouri theory can be simply constructed by promoting the $SO(3,2)$ gauge fields to the corresponding supergroup $OSp(1|4)$ gauge theory. In particular, the gauge potential $\mathcal{A}_\mu^{IJ}$ is now seen as a Lie algebra $osp(1|4)$-valued potential. The corresponding field strength $\mathcal{F}_{\mu\nu}{}^{IJ}$ is given by

$$\mathcal{F}_{\mu\nu}{}^{IJ} = \partial_\mu \mathcal{A}_\nu{}^{IJ} - \partial_\nu \mathcal{A}_\mu{}^{IJ} + \frac{1}{2} f^{IJ}_{JKLM} \mathcal{A}_\mu{}^{JK} \mathcal{A}_\nu{}^{LM}, \tag{17}$$

where f^{IJ}_{JKLM} are the structure constants of the super Lie algebra $osp(1|4)$. The field strength $\mathcal{F}_{\mu\nu}{}^{IJ}$ can be decomposed in three terms corresponding to the three generators $S_{IJ} = (S_{ab}, P_a, Q_i)$ (with $P_a = S_{4a}$) of $osp(1|4)$ as (see [18]-[19] and references therein)

$$\mathcal{F}^{ab}_{\mu\nu} = F^{ab}_{\mu\nu} + \Sigma^{ab}_{\mu\nu} + \Psi^{ab}_{\mu\nu}, \quad \mathcal{F}^{i}_{\mu\nu} = F^{i}_{\mu\nu} + \Sigma^{i}_{\mu\nu}, \quad \mathcal{F}^{a}_{\mu\nu} = F^{a}_{\mu\nu} + \Sigma^{a}_{\mu\nu}, \tag{18}$$

where

$$F_{\mu\nu}{}^{ab} = \partial_\mu A_\nu{}^{ab} - \partial_\nu A_\mu{}^{ab} + \frac{1}{2} f^{ab}_{cdef} A_\mu{}^{cd} A_\nu{}^{ef}, \tag{19}$$

$$\Sigma^{ab}_{\mu\nu} = 2 f^{ab}_{4c4d} A_\mu{}^{4c} A_\nu{}^{4d}, \tag{20}$$

$$\Psi^{ab}_{\mu\nu} = \frac{1}{2} f^{ab}_{ij} A_\mu{}^{i} A_\nu{}^{j}, \tag{21}$$

$$\Sigma^{a}_{\mu\nu} = \frac{1}{2} f^{4a}_{ij} A_\mu{}^{i} A_\nu{}^{j}, \tag{22}$$

$$\Sigma^{i}_{\mu\nu} = f^{i}_{4aj} (A_\mu{}^{4a} A_\nu{}^{j} - A_\nu{}^{4a} A_\mu{}^{j}), \tag{23}$$

and

$$F_{\mu\nu}{}^{i} = \partial_\mu A_\nu{}^{i} - \partial_\nu A^{i}_\mu + \frac{1}{2} f^{i}_{cdj} (A_\mu{}^{cd} A_\nu{}^{j} - A_\nu{}^{cd} A_\mu{}^{j}). \tag{24}$$

Consider the action [10]

$$S = \int d^4x \varepsilon^{\mu\nu\rho\sigma} \mathcal{F}_{\mu\nu}{}^{A} \mathcal{F}_{\rho\sigma}{}^{B} g_{AB}, \tag{25}$$

where g_{AB} is the invariant diagonal metric in $osp(1|4)$ and it is defined by

$$g_{AB} = \begin{pmatrix} \varepsilon_{abcd} & 0 \\ 0 & (C\Gamma_5)_{ij} \end{pmatrix}, \tag{26}$$

where $\Gamma_5 = i\Gamma_0\Gamma_1\Gamma_2\Gamma_3$ and Γ_μ are the Dirac matrices satisfying the Clifford algebra $\{\Gamma_\mu, \Gamma_v\} = -2\eta_{\mu v}$, with $\eta_{\mu v} = diag(-1, 1, 1, 1)$. In terms of its metric components the above-mentioned action can be written as [10], [19]

$$S = \int d^4x \varepsilon^{\mu\nu\rho\sigma} \Big(\mathcal{F}_{\mu\nu}^{\ ab} \mathcal{F}_{\rho\sigma}^{\ cd} \varepsilon_{abcd} + \mathcal{F}_{\mu\nu}^{\ i} \mathcal{F}_{\rho\sigma}^{\ j} (C\Gamma_5)_{ij} \Big). \tag{27}$$

By identifying $A_\mu^{\ ab} \equiv \omega_\mu^{\ ab}$ with the spin connection, $A_\mu^{\ 4a} \equiv e_\mu^{\ a}$ with the tetrad and $A_\mu^{\ i} \equiv \psi_\mu^{\ i}$ with the gravitino field we discover that the action (27) results in the gauge theory of $N = 1$ supergravity (see Ref. [10] for details).

4- Superfield formalism of self-dual MacDowell-Mansouri

In order to obtain the self-dual part of (27), we shall combine the superfield formalism with the internal supergroup $OSp(1|4)$. Let us consider the supersymmetric field-strengths [11]

$$W_{(\alpha)} = -i\lambda_{(\alpha)} + \theta_{(\alpha)} D - \sigma^{\mu v\ (\beta)}_{\ \ (\alpha)} \theta_{(\beta)} \mathcal{F}_{\mu\nu} + \theta^2 \sigma^\mu_{(\alpha\dot{\beta})} \nabla_\mu \bar{\lambda}^{(\dot{\beta})}, \tag{28}$$

$$W^{(\alpha)} = -i\lambda^{(\alpha)} + \theta^{(\alpha)} D + \theta^{(\beta)} \sigma^{\mu\nu\ (\alpha)}_{\ \ (\beta)} \mathcal{F}_{\mu\nu} - \theta^2 \bar{\sigma}^{\mu(\dot{\beta}\alpha)} \nabla_\mu \bar{\lambda}_{(\dot{\beta})}, \tag{29}$$

$$\bar{W}^{(\dot{\alpha})} = i\bar{\lambda}^{(\dot{\alpha})} + \bar{\theta}^{(\dot{\alpha})} D + \bar{\sigma}^{\mu\nu(\dot{\alpha})}_{\qquad (\dot{\beta})} \bar{\theta}^{(\dot{\beta})} \mathcal{F}_{\mu\nu} - \bar{\theta}^2 \bar{\sigma}^{\mu(\dot{\alpha}\alpha)} \nabla_\mu \lambda_{(\alpha)}, \tag{30}$$

and

$$\bar{W}_{(\dot{\alpha})} = i\bar{\lambda}_{(\dot{\alpha})} + \bar{\theta}_{(\dot{\alpha})} D - \bar{\theta}_{(\dot{\beta})} \bar{\sigma}^{\mu\nu(\dot{\beta})}_{\qquad (\dot{\alpha})} \mathcal{F}_{\mu\nu} + \bar{\theta}^2 \sigma^\mu_{(\alpha\dot{\alpha})} \nabla_\mu \lambda^{(\alpha)}, \tag{31}$$

where $(\alpha), (\beta)$ *etc.* are spinor indices, while μ, ν *etc* are space-time indices. (At this moment, we closely follow the References in [11].) Using these expressions we find

$$W^{(\alpha)} W_{(\alpha)} \mid_{\theta\theta} = -\frac{1}{2} \mathcal{F}^{\mu\nu} \mathcal{F}_{\mu\nu} - \frac{i}{4} \varepsilon^{\mu\nu\rho\sigma} \mathcal{F}_{\mu\nu} \mathcal{F}_{\rho\sigma} - 2i\lambda^{(\alpha)} \sigma^\mu_{(\alpha\dot{\beta})} \nabla_\mu \bar{\lambda}^{(\dot{\beta})} + D^2, \tag{32}$$

and as well as

$$\bar{W}_{(\dot{\alpha})} \bar{W}^{(\dot{\alpha})} \mid_{\bar{\theta}\bar{\theta}} = -\frac{1}{2} \mathcal{F}^{\mu\nu} \mathcal{F}_{\mu\nu} + \frac{i}{4} \varepsilon^{\mu\nu\rho\sigma} \mathcal{F}_{\mu\nu} \mathcal{F}_{\rho\sigma} - 2i\lambda^{(\alpha)} \sigma^\mu_{(\alpha\dot{\beta})} \nabla_\mu \bar{\lambda}^{(\dot{\beta})} + D^2 + \nabla_\mu J^\mu, \tag{33}$$

where $J^\mu = 2i\lambda\sigma^\mu\bar{\lambda}$.

Therefore, we get

$$L = \frac{1}{4}(W^{(\alpha)}W_{(\alpha)} \mid_{\theta\theta} + \bar{W}_{(\dot{\alpha})}\bar{W}^{(\dot{\alpha})} \mid_{\bar{\theta}\bar{\theta}}) = -\frac{1}{4}\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu} - i\lambda^{(\alpha)}\sigma^\mu_{(\alpha\dot{\beta})}\nabla_\mu\bar{\lambda}^{(\dot{\beta})} + \frac{1}{2}D^2, \tag{34}$$

and also

$$L_t = \frac{i}{4}(W^{(\alpha)}W_{(\alpha)} \mid_{\theta\theta} - \bar{W}_{(\dot{\alpha})}\bar{W}^{(\dot{\alpha})} \mid_{\bar{\theta}\bar{\theta}}) = \frac{1}{8}\varepsilon^{\mu\nu\rho\sigma}\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}. \tag{35}$$

Here, we dropped the surface term $\nabla_\mu J^\mu$.

If we introduce the definition

$${}^{\pm}\mathcal{F}_{\mu\nu}{}^{A} = \frac{1}{2}{}^{\pm}\mathcal{B}^A_B\mathcal{F}^B_{\mu\nu}, \tag{36}$$

where ${}^{\pm}\mathcal{B}^A_B$ is given by

$$\begin{aligned} {}^{\pm}\mathcal{B}^A_B &= \begin{pmatrix} {}^{\pm}B^{ab}_{cd} & 0 \\ 0 & {}^{\pm}B^i_j \end{pmatrix} \\ &= \begin{pmatrix} \delta^{ab}_{cd} \pm i\varepsilon^{ab}_{cd} & 0 \\ 0 & \frac{1}{2}(1 \pm \Gamma_5)^i_j \end{pmatrix}. \end{aligned} \tag{37}$$

It is not difficult to see that the self-dual sector of L_t is given by

$${}^{+}L_t = \frac{i}{4}({}^{+}W^{(\alpha)+}W_{(\alpha)} \mid_{\theta\theta} - {}^{+}\bar{W}^{+}_{(\dot{\alpha})}\bar{W}^{(\dot{\alpha})} \mid_{\bar{\theta}\bar{\theta}}) = \frac{1}{8}\varepsilon^{\mu\nu\rho\sigma+}\mathcal{F}^A_{\mu\nu}{}^{+}\mathcal{F}^B_{\mu\nu}g_{AB}, \tag{38}$$

where g_{AB} is the metric associated with the superalgebra $osp(1|4)$ and it is given by the formula (26).

Using (26) we discover that (38) can be written as:

$${}^{+}L_t = \frac{1}{8}\varepsilon^{\mu\nu\rho\sigma+}({}^{+}\mathcal{F}^{ab}_{\mu\nu}{}^{+}\mathcal{F}^{cd}_{\rho\sigma}\varepsilon_{abcd} + {}^{+}\mathcal{F}_{\mu\nu}{}^{i}\,{}^{+}\mathcal{F}^j_{\rho\sigma}(C\Gamma_5)_{ij}). \tag{39}$$

In this expression, we recognize the Lagrangian proposed by Nieto, Socorro and Obregón [10], which generalizes the corresponding Lagrangian (27) to the self-dual case.

A generalization of (39) can be achieved by means of the action [19]

$$S = \int d^4x\varepsilon^{\mu\nu\rho\sigma}\Big({}^{+}\tau^{+}\mathcal{F}^A_{\mu\nu}{}^{+}\mathcal{F}^B_{\rho\sigma}g_{AB} - {}^{-}\tau^{-}\mathcal{F}^A_{\mu\nu}{}^{-}\mathcal{F}^B_{\rho\sigma}g_{AB}\Big), \tag{40}$$

which considers both the self-dual sector as well as the antiself-dual sector. In this part, ${}^{+}\tau$ and ${}^{-}\tau$ are constant coupling parameters. The action (40) may provide the starting point for studying the supersymmetric case of a $S-$duality program for gravity, which was initiated in [19] for the MacDowell-Mansouri pure gauge theory.

5. Toward an eight dimensional superfield formalism of a self-dual supergravity a la MacDowell-Mansouri

Now, we shall consider that it is possible to extend the procedure of section 3 to eight-dimensions. Observe first the important role played by the $\varepsilon-$symbols $\varepsilon^{\mu\nu\rho\sigma}$, ε^{abcd} and Γ_5 in the expressions (37) and (38). In fact, the symbol $\varepsilon^{\mu\nu\rho\sigma}$ determines that the dimensionality of the spacetime should be four; ε^{abcd} is $SO(3,1)-$invariant object and is used to define self-duality (antiself-duality) in the bosonic sector, while $\Gamma_5 = i\Gamma_0\Gamma_1\Gamma_2\Gamma_3$ is directly related to the existence of Dirac matrices satisfying a Clifford algebra in four dimensions and determines the self-dual (antiself-dual) in the fermionic sector. In general, since in any higher dimensional spacetime the field strength $\mathcal{F}^{A}_{\mu\nu}$ is a two-form matrix , it seems a difficult task to consider its self-dual and antiself-dual sectors other than four dimensions. Nevertheless, in the case of spin 1 it has been proved that an interesting possibility arises in eight dimensions [20]. The idea can be traced back to the observation that ε^{0bcd} is linked to an exceptional algebra: the quaternionic algebra. So, if one considers the octonion structure, which is also an exceptional division algebra, one should be able to consider the $\eta-$symbols $\eta^{\mu\nu\rho\sigma}$, η^{abcd} and $\Gamma_9 = i\Gamma_0 \cdots \Gamma_7$. In fact, some progress in using the $\eta-$symbols has been achieved in the self-dual Yang-Mills field case [20] (see also [21]) and in the bosonic Ashtekar formalism in eight dimensions [13]. The key formula for applying the $\eta-$symbols in these cases is

$$\eta_{\mu\nu\alpha\beta}\eta^{\tau\sigma\alpha\beta} = 6\delta^{\tau\sigma}_{\mu\nu} + 4\eta^{\tau\sigma}_{\mu\nu}, \tag{41}$$

which, in contrast to the analogue $\varepsilon-$symbol expression

$$\varepsilon_{\mu\nu\alpha\beta}\varepsilon^{\tau\sigma\alpha\beta} = 2\delta^{\tau\sigma}_{\mu\nu}, \tag{42}$$

leads to self-duality of the field strength ${}^{+}\mathcal{F}^{ab}_{\mu\nu} = \frac{1}{2}(\delta^{ab}_{cd} + \eta^{ab}_{cd})\mathcal{F}^{ab}_{\mu\nu}$ in the form,

$${}^{*+}\mathcal{F}^{ab}_{\mu\nu} = 3{}^{+}\mathcal{F}^{ab}_{\mu\nu}. \tag{43}$$

Thus, except for a numerical factor ${}^{+}\mathcal{F}^{ab}_{\mu\nu}$ satisfies the usual self-dual relation.

In general, in a superspace in eight dimensions $(x^\mu, \theta_{(A)})$, with a $\mu = 1, ..., 8$ and $(A) = 1, ..., 16$, a vector superfield can be written as [22]

$$V_J = \sum_{(A_i)=1}^{16} \theta^{(A_1)} \cdots \theta^{(A_{16})} V_{J(A_1)\cdots(A_{16})}(x), \tag{44}$$

where $\theta_{(A)}$ are Grassmann anticommuting variables. In order to write the corresponding superfield strengths in eight dimensions, $W^{(\alpha)}$ and $\bar{W}_{(\dot{\alpha})}$, it turns out to be convenient to work on the Euclidean sector and subsequent ones to perform a Wick rotation. This allows to consider the $SO(8)$ matrices Γ_μ, satisfying the Clifford algebra $\{\Gamma_\mu, \Gamma_v\} = 2\delta_{\mu v}$, in the $Spin(7)$ representation

$$\begin{gathered} \Gamma_{\hat{a}} = \begin{pmatrix} 0 & -i\gamma_{\hat{a}} \\ i\gamma_{\hat{a}} & 0 \end{pmatrix}, \\ \Gamma_8 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \ \ \Gamma_9 \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \end{gathered} \tag{45}$$

where $(\gamma_{\hat{a}})_{(\alpha\beta)}$ $(\alpha, \beta = 1, ..., 8$ and $\hat{a} = 1, ..., 7)$ are antisymmetric $Spin(7)$ matrices defined by

$$\begin{gathered} (\gamma_{\hat{a}})_{(\hat{b}8)} = i\delta_{\hat{a}\hat{b}}, \\ (\gamma_{\hat{a}})_{(\hat{b}\hat{c})} = iC_{\hat{a}\hat{b}\hat{c}}. \end{gathered} \tag{46}$$

Here, $C_{\hat{a}\hat{b}\hat{c}}$ are the totally antisymmetric octonion structure constants, which are related to the dual totally antisymmetric octonionic tensor η_{abcd} in form

$$C_{\hat{a}\hat{b}\hat{c}} = \eta_{8\hat{a}\hat{b}\hat{c}} \tag{47}$$

and

$${}^{\pm}F_{\hat{a}\hat{b}\hat{c}\hat{d}} \equiv C_{\hat{a}\hat{b}\hat{e}} C_{\hat{c}\hat{d}}^{\ \ \hat{e}} = \delta_{\hat{a}\hat{c}}\delta_{\hat{b}\hat{d}} - \delta_{\hat{a}\hat{d}}\delta_{\hat{b}\hat{c}} \pm \eta_{\hat{a}\hat{b}\hat{c}\hat{d}}. \tag{48}$$

Observe that, since $\eta_{\hat{a}\hat{b}\hat{c}\hat{d}}$ is totally antisymmetric, both solutions in (48) lead to the expression

$$C_{\hat{a}\hat{b}\hat{e}} C_{\hat{c}\hat{d}}^{\ \ \hat{e}} + C_{\hat{c}\hat{b}\hat{e}} C_{\hat{a}\hat{d}}^{\ \ \hat{e}} = 2\delta_{\hat{a}\hat{c}}\delta_{\hat{b}\hat{d}} - \delta_{\hat{a}\hat{d}}\delta_{\hat{b}\hat{c}} - \delta_{\hat{c}\hat{d}}\delta_{\hat{b}\hat{a}}, \tag{49}$$

which determines the octonions as a normed algebra (see [23] and Refs. therein).

For later computation we note that if one assumes the ${}^{+}F_{\hat{a}\hat{b}\hat{c}\hat{d}}$ solution then we obtain the identity

$$\eta_{\mu\nu\alpha\beta}\eta^{\tau\sigma\alpha\beta} = 6\delta^{\tau\sigma}_{\mu\nu} + 4\eta^{\tau\sigma}_{\mu\nu}, \tag{50}$$

while if one assumes the ${}^{-}F_{\hat{a}\hat{b}\hat{c}\hat{d}}$ solution we get

$$\eta_{\mu\nu\alpha\beta}\eta^{\tau\sigma\alpha\beta} = 6\delta^{\tau\sigma}_{\mu\nu} - 4\eta^{\tau\sigma}_{\mu\nu}. \tag{51}$$

In the representation (45)-(46), the generators $\Gamma_{ab} = \frac{1}{2}[\Gamma_a, \Gamma_b]$ of $SO(8)$ become

$$\begin{aligned} \Gamma_{\hat{a}\hat{b}} &= \begin{pmatrix} (\gamma_{\hat{a}\hat{b}})_{(\alpha\beta)} & 0 \\ 0 & (\gamma_{\hat{a}\hat{b}})_{(\alpha\beta)} \end{pmatrix}, \\ \Gamma_{\hat{a}8} &= \begin{pmatrix} -i(\gamma_{\hat{a}})_{(\alpha\beta)} & 0 \\ 0 & i(\gamma_{\hat{a}})_{(\alpha\beta)} \end{pmatrix}. \end{aligned} \tag{52}$$

Here, $(\gamma_{\hat{a}\hat{b}})_{(\alpha\beta)}$ are the antisymmetric $Spin(7)$ generators,

$$\begin{aligned} &(\gamma_{\hat{a}\hat{b}})_{(\hat{c}8)} = C_{\hat{a}\hat{b}\hat{c}}, \\ &(\gamma_{\hat{a}\hat{b}})_{(\hat{c}\hat{d})} = \delta_{\hat{a}\hat{c}}\delta_{\hat{b}\hat{d}} - \delta_{\hat{a}\hat{d}}\delta_{\hat{b}\hat{c}} - \eta_{\hat{a}\hat{b}\hat{c}\hat{d}}. \end{aligned} \tag{53}$$

In order to split $\theta^{(A)}$ into two Weyl spinors $\theta^{(\alpha)}$ and $\bar{\theta}^{(\dot{\alpha})}$ one can use Γ_9 matrix. Accordingly, using (45) we can write

$$\Gamma_a = \begin{pmatrix} 0 & \sigma_a \\ \bar{\sigma}_a & 0 \end{pmatrix}, \tag{54}$$

where $\sigma_a = (-i\gamma_{\hat{a}}, 1)$ and $\bar{\sigma}_a = (i\gamma_{\hat{a}}, 1)$. Thus, we may introduce the corresponding $Spin(7)$ generators

$$\sigma^{ab\,(\beta)}_{\quad(\alpha)} = \frac{1}{2}(\sigma^a\bar{\sigma}^b - \sigma^b\bar{\sigma}^a)^{(\beta)}_{(\alpha)}, \tag{55}$$

and

$$\bar{\sigma}^{ab\,(\beta)}_{\quad(\alpha)} = \frac{1}{2}(\bar{\sigma}^a\sigma^b - \bar{\sigma}^b\sigma^a)^{(\beta)}_{(\alpha)}. \tag{56}$$

In fact, using the ${}^{+}F_{\hat{a}\hat{b}\hat{c}\hat{d}}$ solution given in (48) one may verify that

$$\begin{aligned} &\sigma_{\hat{a}\hat{b}(\hat{c}8)} = C_{\hat{a}\hat{b}\hat{c}}, \\ &\sigma_{\hat{a}\hat{b}(\hat{c}\hat{d})} = \delta_{\hat{a}\hat{c}}\delta_{\hat{b}\hat{d}} - \delta_{\hat{a}\hat{d}}\delta_{\hat{b}\hat{c}} - \eta_{\hat{a}\hat{b}\hat{c}\hat{d}}, \end{aligned} \tag{57}$$

and also

$$\sigma_{\hat{a}8\hat{c}8} = \delta_{\hat{a}\hat{c}},$$
$$\sigma_{\hat{a}8\hat{b}\hat{c}} = C_{\hat{a}\hat{b}\hat{c}}. \tag{58}$$

We also get

$$\bar{\sigma}_{\hat{a}\hat{b}(\hat{c}8)} = C_{\hat{a}\hat{b}\hat{c}},$$
$$\bar{\sigma}_{\hat{a}\hat{b}(\hat{c}\hat{d})} = \delta_{\hat{a}\hat{c}}\delta_{\hat{b}\hat{d}} - \delta_{\hat{a}\hat{d}}\delta_{\hat{b}\hat{c}} - \eta_{\hat{a}\hat{b}\hat{c}\hat{d}}, \tag{59}$$

and

$$\bar{\sigma}_{\hat{a}8\hat{c}8} = -\delta_{\hat{a}\hat{c}},$$
$$\bar{\sigma}_{\hat{a}8\hat{b}\hat{c}} = -C_{\hat{a}\hat{b}\hat{c}}. \tag{60}$$

It is not difficult to see that the expressions (57) and (58) can be unified in the form

$$\sigma_{\mu\nu(\alpha\beta)} = \delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha} - \eta_{\mu\nu\alpha\beta}. \tag{61}$$

If we use the identity (50) we find

$$\sigma_{\mu\nu(\lambda\tau)}\sigma_{\alpha\beta}^{(\lambda\tau)} = 8(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}). \tag{62}$$

From the discussion of section 3, one should expect that in the computation of $W^{(\alpha)}W_{(\alpha)}\mid_{\theta\theta}$ the expression (62) leads to the analogue in eight dimensions of the term $-\frac{1}{2}\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}$ rather than the analogue of the term $-\frac{1}{2}\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu} - \frac{i}{4}\varepsilon^{\mu\nu\rho\sigma}\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}$. Moreover, we find that it seems to be no way to unified (59) and (60) in just one expression for $\bar{\sigma}_{\mu\nu(\alpha\beta)}$ as in the case of $\sigma_{\mu\nu(\alpha\beta)}$. These two observations lead us to consider an alternative prescription.

Consider the quantities

$$\Theta_{\mu\nu\alpha\beta} = \delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha} + \eta_{\mu\nu\alpha\beta} \tag{63}$$

and

$$\bar{\Theta}_{\mu\nu\alpha\beta} = \delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha} - \eta_{\mu\nu\alpha\beta}, \tag{64}$$

which are related to the ${}^{+}F_{\hat{a}\hat{b}\hat{c}\hat{d}}$ and ${}^{-}F_{\hat{a}\hat{b}\hat{c}\hat{d}}$ solutions respectively. From the ${}^{+}F_{\hat{a}\hat{b}\hat{c}\hat{d}}$ solution we find that $\Theta_{\mu\nu\alpha\beta}$ is a self-dual projector. In fact, using (50) we obtain

$$\Theta_{\mu\nu\tau\lambda}\Theta_{\alpha\beta}^{\tau\lambda} = 8\Theta_{\mu\nu\alpha\beta} \tag{65}$$

and

$$\frac{1}{2}\eta_{\mu\nu\tau\lambda}\Theta^{\tau\lambda}_{\alpha\beta} = 3\Theta_{\mu\nu\alpha\beta}. \tag{66}$$

As it has been emphasized in Ref. [24] the object $\Theta_{\mu\nu\tau\lambda}$ projects any antisymmetric second rank tensor onto its self-dual parts **7**, according to the decomposition **28**=**7**$\oplus$**21** of the adjoint representation of $SO(8) \sim SO(8)/Spin(7) \oplus Spin(7)$.

Similarly, using the formula (51) which corresponds to the ${}^{-}F_{\hat{a}\hat{b}\hat{c}\hat{d}}$ solution, we find that $\bar{\Theta}_{\mu\nu\tau\lambda}$ satisfies

$$\bar{\Theta}_{\mu\nu\tau\lambda}\bar{\Theta}^{\tau\lambda}_{\alpha\beta} = 8\bar{\Theta}_{\mu\nu\alpha\beta} \tag{67}$$

and

$$\frac{1}{2}\eta_{\mu\nu\tau\lambda}\bar{\Theta}^{\tau\lambda}_{\alpha\beta} = -3\bar{\Theta}_{\mu\nu\alpha\beta}. \tag{68}$$

Therefore $\bar{\Theta}_{\mu\nu\tau\lambda}$ is an antiself-dual projector operator.

Thus, one should expect that in the computation of $W^{(\alpha)}W_{(\alpha)}\mid_{\theta\theta}$ the expression (65) may lead to the analogue in eight dimensions of the combination $-\frac{1}{2}\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu} - \frac{i}{4}\varepsilon^{\mu\nu\rho\sigma}\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}$ in four dimensions. This means that, for our purpose, the self-dual and the antiself-dual splitting associated with the group $SO(8)$ provide a better alternative than the Weyl splitting specified in (54).

With these tools at hand, one may proceed as in four dimensions, that is, one may derive presumably from (44) the eight dimensional fields strengths

$$W_{(\alpha)} = -i\lambda_{(\alpha)} + \theta_{(\alpha)}D - \Theta^{\mu\nu}{}_{(\alpha)}{}^{(\beta)}\theta_{(\beta)}\mathcal{F}_{\mu\nu} + \theta^2\sigma^{\mu}_{(\alpha\dot{\beta})}\nabla_{\mu}\bar{\lambda}^{(\dot{\beta})} + ..., \tag{69}$$

$$W^{(\alpha)} = -i\lambda^{(\alpha)} + \theta^{(\alpha)}D + \theta^{(\beta)}\Theta^{\mu\nu}{}_{(\beta)}{}^{(\alpha)}\mathcal{F}_{\mu\nu} - \theta^2\bar{\sigma}^{\mu(\dot{\beta}\alpha)}\nabla_{\mu}\bar{\lambda}_{(\dot{\beta})} + ..., \tag{70}$$

$$\bar{W}^{(\dot{\alpha})} = i\bar{\lambda}^{(\dot{\alpha})} + \bar{\theta}^{(\dot{\alpha})}D + \bar{\Theta}^{\mu\nu(\dot{\alpha})}{}_{(\dot{\beta})}\bar{\theta}^{(\dot{\beta})}\mathcal{F}_{\mu\nu} - \bar{\theta}^2\bar{\sigma}^{\mu(\dot{\alpha}\alpha)}\nabla_{\mu}\lambda_{(\alpha)}..., \tag{71}$$

and

$$\bar{W}_{(\dot{\alpha})} = i\bar{\lambda}_{(\dot{\alpha})} + \bar{\theta}_{(\dot{\alpha})}D - \bar{\theta}_{(\dot{\beta})}\bar{\Theta}^{\mu\nu(\dot{\beta})}{}_{(\dot{\alpha})}\mathcal{F}_{\mu\nu} + \bar{\theta}^2\sigma^{\mu}_{(\alpha\dot{\alpha})}\nabla_{\mu}\lambda^{(\alpha)} + ... \tag{72}$$

Here, by convenience, we only wrote the relevant terms in the θ power expansion. The object $\mathcal{F}_{\mu\nu}$ should be seen now as an eight dimensional nonabelian field strength and $\lambda_{(\alpha)}$ as an eight dimensional chiral spinor.

Using (69)-(72) and (65) we find

$$W^{(\alpha)}W_{(\alpha)}\mid_{\theta\theta}=16\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}+8\eta^{\mu\nu\rho\sigma}\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}-2i\lambda^{(\alpha)}\sigma^{\mu}_{(\alpha\dot{\beta})}\nabla_{\mu}\bar{\lambda}^{(\dot{\beta})}+D^{2}. \tag{73}$$

While if we use (67) we get

$$\bar{W}_{(\dot{\alpha})}\bar{W}^{(\dot{\alpha})}\mid_{\bar{\theta}\bar{\theta}}=16\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}-8\eta^{\mu\nu\rho\sigma}\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}-2i\lambda^{(\alpha)}\sigma^{\mu}_{(\alpha\dot{\beta})}\nabla_{\mu}\bar{\lambda}^{(\dot{\beta})}+D^{2}, \tag{74}$$

where we dropped the surface term $\nabla_{\mu}(2i\lambda\sigma^{\mu}\bar{\lambda})$. (It is worth mentioning that (73) corresponds to the Lagrangian discussed in Ref. [24] for the case of $7D$ super Yang-Mills theory.) Therefore, we get

$$L=\frac{1}{4}(W^{(\alpha)}W_{(\alpha)}\mid_{\theta\theta}+\bar{W}_{(\dot{\alpha})}\bar{W}^{(\dot{\alpha})}\mid_{\bar{\theta}\bar{\theta}})=8\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}-i\lambda^{(\alpha)}\sigma^{\mu}_{(\alpha\dot{\beta})}\nabla_{\mu}\bar{\lambda}^{(\dot{\beta})}+\frac{1}{2}D^{2}, \tag{75}$$

and also

$$L_t=\frac{1}{4}(W^{(\alpha)}W_{(\alpha)}\mid_{\theta\theta}-\bar{W}_{(\dot{\alpha})}\bar{W}^{(\dot{\alpha})}\mid_{\bar{\theta}\bar{\theta}})=4\eta^{\mu\nu\rho\sigma}\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}. \tag{76}$$

If we now introduce the definition

$${}^{\pm}\mathcal{F}_{\mu\nu}{}^{A}=\frac{1}{2}{}^{\pm}\mathcal{B}^{A}_{B}\mathcal{F}^{B}_{\mu\nu}, \tag{77}$$

where ${}^{\pm}\mathcal{B}^{A}_{B}$ is given by

$$\begin{aligned}{}^{\pm}\mathcal{B}^{A}_{B}&=\begin{pmatrix}{}^{\pm}B^{ab}_{cd} & 0\\ 0 & {}^{\pm}B^{i}_{j}\end{pmatrix}\\ &=\begin{pmatrix}\delta^{ab}_{cd}\pm\eta^{ab}_{cd} & 0\\ 0 & \frac{1}{2}(1\pm\Gamma_9)^{i}_{j}\end{pmatrix},\end{aligned} \tag{78}$$

we find the Lagrangian

$${}^{+}L_t=\frac{1}{4}({}^{+}W^{(\alpha)+}W_{(\alpha)}\mid_{\theta\theta}-{}^{+}\bar{W}^{+}_{(\dot{\alpha})}\bar{W}^{(\dot{\alpha})}\mid_{\bar{\theta}\bar{\theta}})=4\eta^{\mu\nu\rho\sigma}\;{}^{+}\mathcal{F}^{A}_{\mu\nu}\,{}^{+}\mathcal{F}^{B}_{\mu\nu}g_{AB}, \tag{79}$$

which describes the self-dual sector of L_t in eight dimensions. Of course, in this case the metric g_{AB} should be chosen as

$$g_{AB} = \left(\begin{array}{cc} \eta_{abcd} & 0 \\ 0 & (C\Gamma_9)_{ij} \end{array} \right). \tag{80}$$

Therefore, we have derived a self-dual supergravity Lagrangian in eight dimensions via superfield formalism. It turns out that, the bosonic sector of the Lagrangian (79) is reduced to the Lagrangian proposed in the Ref. [13]. This seems to be an important result in the quest of a supersymmetric Ashtekar formalism in eight dimensions. Moreover, there is an important observation that we need to make in connection with the invariance of Lagrangian (79). In general the ε-symbol is Lorentz invariant in any dimension, but in contrast the η-symbol is only $SO(7)$-invariant in eight dimensions (see Ref. [25]). Therefore, the η-symbol spoils the Lorentz invariance of the Lagrangian ${}^{+}L_t$ given in (79), but maintains a hidden $SO(7)$-invariance. In fact, this is a general phenomenon in field theories involving the η-symbol (see, for instance, Refs. [24] and [27]).

6. Final remarks

It is well known the great importance of the role that a nonabelian field strength plays in the gauge field theories. In a sense, it is the main object in any gauge field action. Mathematically, the nonabelian field strength is recognized as the two-form curvature in a fiber bundle. Surprisingly, the possibilities to make self-dual such a two-form is very restrictive. In fact, independently of the signature of spacetime, self-duality seems to work only in dimensions $1, 2, 4,$ and 8. These dimensions can be recognized as the dimensions associated with the exceptional algebras: real, complex, quaternion and octonion algebras. Of course, in order to make sense of self-duality in any dimension one may introduce p-form gauge fields but this case works only as an abelian theory [26]. If one is thinking about gravity as a nonabelian gauge theory, then one obtains a similar conclusion concerning the dimensions $1, 2, 4,$ and 8. Therefore, from this point of view it seems natural to consider the self-dual supergravity in eight dimensions. And the best way to do this is by using a superfield prescription with an octonion algebra as a background structure. Pursuing this idea we first applied a superfield formalism to self-dual supergravity in four dimensions showing the consistency of the procedure and then we applied similar technique for the case of self-dual supergravity in eight dimensions.

Eight dimensional self-dual supergravity has been previously studied by Nishino and Rajpoot [27]-[29] (see also Refs. [24] and [30]-[31]) whom based their motivation on twelve dimensional supergravity [32]. However, in this

formalism the superfield prescription is not used and there is not any guide for a connection with the Ashtekar formalism. Nevertheless, it seems interesting for further research to find a connection between the works of these authors and the formalism presented here. Eleven-dimensional supergravity in light-cone superspace [33] has been studied recently. The formalism in this work is based on the decomposition $SO(9) \supset SO(2) \times SO(7)$. The $SO(7)$ subgroup is used on the superfield formalism without using the octonion structure. Perhaps, our work may be useful in this direction.

It has been noted [34] that the $\varepsilon-$symbol $\varepsilon^{\mu\nu\alpha\beta}$ is a chirotope [35] and that $\eta^{\mu\nu\rho\sigma}$ is related to the Fano matroid [36]. This means that the Lagrangians (76) and (79) contain information of the oriented matroid theory [35]. Therefore, the present work suggested that, besides all the relations between matroids and different aspects of $M-$theory, including gravity [37], Chern-Simons theory [38] and String theory [39], Matroid theory should be connected with the superfield formalism.

Starting with the supersymmetric Lagrangian in eight dimensions given in (79) one may be interested in studying self-duality structure in supersymmetric $10+2-$dimensional model via the two projections $10+2 \to (1+3)+(1+7)$ and $10+2 \to (2+2)+(0+8)$. If this program goes on the right route then we shall become closer to find a connection between the "self-dual" supergravity theory and the supersymmetric Ashtekar formalism, in $10+2-$dimensional spacetime.

Finally, in order to achieve a manifest $SO(3,2)$ symmetry in the context of the bosonic MacDowell-Mansouri formulation Stelle and West [40] introduced the idea of compensator field. In Ref. [41] Vasiliev developed the MacDowell-Mansouri-Stelle-West action for any dimension and in Ref. [42] the compensator approach was applied to the study of $N = 1$ supergravity in four dimensions as a gauge theory of $OSp(1|4)$. Thus, it seems interesting to see whether the results of the present work may be combined with the idea of a compensator approach in higher dimensional supergravity theories.

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